Revealed Preference Dimension via Matrix Sign Rank

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Revealed Preference

Alice wants to buy some fruit.

She sees an apple for \$1, and an orange for \$1.50, and decides to buy the orange.

What can we conclude?

Alice has revealed that she prefers oranges to apples, since she was willing to pay more for an orange.

$$\text{“orange} \succ \text{apple”}$$

Observing Alice's purchases, we can determine her relative ordering over outcomes $\Longrightarrow$ **Revealed Preference**.

Classical economic theory, [Samuelson, '38]

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Preference Graphs

[Afriat, '67]: Behaviour "consistent" if and only if no cycles

$$\text{orange} \succ \text{apple} \succ \left\{ \begin{array}{c} \text{grape and} \\ \text{1/2 banana} \end{array} \right\} \succ \text{banana} \succ \text{orange}$$

View it as a graph

Revealed-Preference Graphs

There is a node for each possible bundle, and whenever a purchase is made, add an arc pointing from the chosen bundle to all cheaper.

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Motivation: Detecting Untruthful Behaviour

Where is this used?

Heuristic means to enforce truthfulness in repeated settings *e.g.*

Ascending Combinatorial Auctions

Buyers want subset of n items. (1) Mechanism sets price for each, (2) buyers choose favourite bundle. (3) Increase prices and repeat until no conflicts.

Bidding/Activity Rules

Maintain preference graph over subsets of items, disallow cycles.

Weaker Rules: Sometimes useful to weaken, *e.g.*

- Delete $\leq k$ nodes to get DAG [Houtman, Maks, '85]
- Delete small-weight edges to get DAG [Afriat, '73]
- Etc.

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Motivation: Computational Problems

Bidding rules often standard graph properties of preference graphs

$\implies$ well-studied computational problems, some hard.

What if the graphs are not general? *e.g.* Small number of items

Geometric Preference Graphs

Consider a commodity market with budget-constrained buyers. Let $\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_n \in \mathbb{R}^d_{\geq 0}$ be vectors of item prices (d items).

Fix one buyer, and say chooses bundle $\mathbf{x}_t$ when prices $\mathbf{p}_t$ for all $t = 1, 2, \ldots, n$.

Have $\mathbf{x}_t \succ \mathbf{x}_s$ if $\langle \mathbf{p}_t, \mathbf{x}_t \rangle \geq \langle \mathbf{p}_t, \mathbf{x}_s \rangle$. Preference graph defined as usual.

Question: For d fixed, which preference graphs are possible?

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Revealed-Preference Dimension

Structural Converse question:

RP-Dimension

Given a directed graph G on n vertices, what is the minimum d such that there exist $\mathbf{p}_1, \mathbf{x}_1, \ldots, \mathbf{p}_n, \mathbf{x}_n \in \mathbb{R}^d_{\geq 0}$ where $(i, j) \in G$ if and only if $\langle \mathbf{p}_i, \mathbf{x}_i \rangle > \langle \mathbf{p}_i, \mathbf{x}_j \rangle$.

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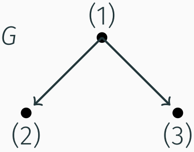
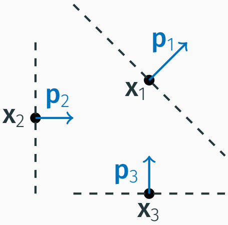
Graph G





Matrix Sign Rank

We answer the question in terms of the Matrix Sign Rank of a modified adjacency matrix.

Matrix Sign-Rank

Given a sign-matrix $S \in \{+1, -1, 0\}^{n \times m}$, what is the least-rank matrix $M \in \mathbb{R}^{n \times m}$ such that $sign(M_{ij}) = S_{ij}$?

Consider *e.g.* the following with sign-rank 3:

$$\begin{bmatrix} 0 & + & + & + \\ - & 0 & - & + \\ - & - & 0 & + \\ - & - & - & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 2 & 4 \\ -2 & 0 & -1 & 1 \\ -2 & -1 & 0 & 1 \\ -4 & -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -2 & -2 \\ 1 & -1 & -2 \\ 1 & -2 & -1 \\ 0 & -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

So rank ≤ 3, can show no two columns can span the rest (signs).

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Why Sign Rank?

We want to show RP-Dimension(G) $=$ Sign-Rank("$M(G)$").

Well-studied: Many results [AFR85, Mnëv89, RS10, BK15, ...], including $O(n/\log n)$-factor approx. [AMY16]

Geometric: Also good geometric interpretation: Low-rank matrix gives low-dimensional points in space

$$\begin{bmatrix} n \times m \end{bmatrix} = \underbrace{\begin{bmatrix} n \times r \end{bmatrix}}_{\text{points}} \cdot \underbrace{\begin{bmatrix} r \times m \end{bmatrix}}_{\text{hyperplanes}}$$

Hardness?

Computing sign rank for $\{+,-,0\}^{m \times n}$ is $\exists\mathbb{R}$-complete in general, but not for our special case. Only known to be NP-hard. ($\{+,-\}^{m \times n}$)

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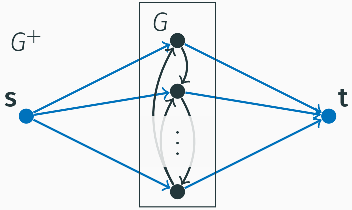
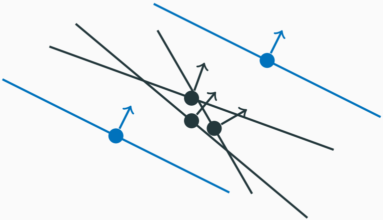
Theorem 1: Reduction to Sign Rank

The idea is as follows: To guarantee $\mathbf{p}_i \in \mathbb{R}^d_{\geq 0}$, add 2 points





This allows transformation to get positive prices. Then set,

$$M(G^+)_{ij} = \begin{cases} 0 & \text{if } i = j \\ +1 & \text{if } (i,j) \in G^+ \\ -1 & \text{otherwise} \end{cases}$$

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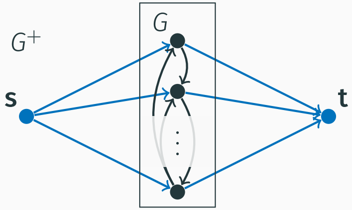

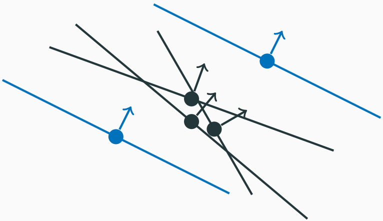

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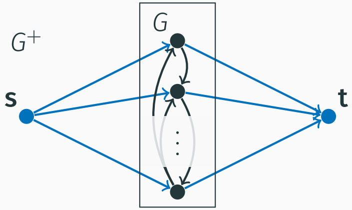

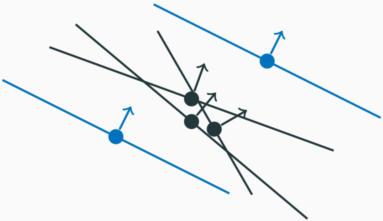

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Proof of Theorem 1.

Proof:

$$\underbrace{\left[\begin{array}{ccc|c} \leftarrow & \mathbf{p}_1 & \rightarrow & \langle \mathbf{p}_1, \mathbf{x}_1 \rangle \\ \leftarrow & \mathbf{p}_2 & \rightarrow & \langle \mathbf{p}_2, \mathbf{x}_2 \rangle \\ & \vdots & & \vdots \\ \leftarrow & \mathbf{p}_n & \rightarrow & \langle \mathbf{p}_n, \mathbf{x}_n \rangle \end{array}\right]}_{n \times (d+1)} \cdot \underbrace{\left[\begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ -\mathbf{x}_1 & -\mathbf{x}_2 & \cdots & -\mathbf{x}_n \\ \downarrow & \downarrow & & \downarrow \\ \hline 1 & 1 & \cdots & 1 \end{array}\right]}_{(d+1) \times n}$$

(i, j)-th entry is $\langle \mathbf{p}_i, \mathbf{x}_i \rangle - \langle \mathbf{p}_i, \mathbf{x}_j \rangle$, positive $\iff$ arc. $\square$

Theorem 2: Bounds by Poset Order Dimension

Special Case

What if G is a partial order? (*i.e.* Deduced transitive preferences)

Then order dimension is a good bound

Poset Order Dimension

Let P be a partial order, and $L_1, \ldots, L_k$ be a *minimum* collection of total (linear) orders, such that $P = L_1 \cap \cdots \cap L_k$. Then $dim(P) = k$.

This is equivalent to dominance relation:
Collection of vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^k$, with $\mathbf{x}_i \succ \mathbf{x}_j \iff (\mathbf{x}_i)_\ell \geq (\mathbf{x}_j)_\ell \forall \ell$

The k total orders are just the orders in each dimension.

Theorem 2, 3.

If G is a poset di-graph, $dim(G) = k$, then RP-DIMENSION$(G) \leq k$.
Also, $\min\{k, 3\} \leq$ RP-DIMENSION(G). *Lower bound is tight.*

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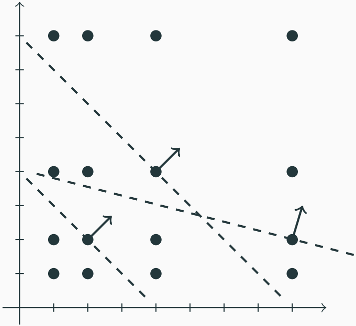

□

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Questions?